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LETTER TO THE EDITOR

All quantum group structures on the supergroup $GL(1 \mid 1)$

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Abstract. All quantum group structures are found on the supergroup $GL(1 \mid 1)$. These structures are described by two two-parameter families, $GL_{q,Q}(1 \mid 1)$ and $GL_{h_1,h_2}(1 \mid 1)$. Each family possesses a central multiplicative quantum superdeterminant, which allows one to define the quantum supergroups $SL_{q,Q}(1 \mid 1)$ and $SL_{h_1,h_2}(1 \mid 1)$.

Quantum analogues of a given classical object are, in general, not unique. When the object is a group (or a supergroup), the variety of possible quantum analogues is, as a rule, very large. There are two exceptions to the rule when the group is a matrix one: the group $GL(2)$, and the supergroup $GL(1 \mid 1)$. In the former case, there exist precisely two quantizations of $GL(2)$ which can be restricted to $SL(2)$ (by virtue of having the quantum determinant to be central), namely $GL_q(2)$ and $GL_h(2)$ (Kupershmidt 1992). Let us now look at the supergroup $GL(1 \mid 1)$.

We start off with the quasiclassical description. In other words, let us first determine all Lie–Poisson structures on the supergroup $GL(1 \mid 1)$. One can show that, for the general case of $GL(n \mid m)$, all such structures arise as Poisson symmetries of a pair of Poisson superplanes, $V^{n \mid m}$ and $\overline{V}^{(n \mid m)}$ say, of opposite Z_2-gradings. For the case at hand, we have two $(1 \mid 1)$-dimensional superplanes, with coordinates which we denote (x, ξ) and (η, y) respectively. From Z_2-dimensional considerations, the most general quadratic Poisson brackets on these planes are given by the formulae

$$\{x, \xi\} = p_1 x\xi \qquad \{\xi, \xi\} = p_2 x^2 \tag{1a}$$

$$\{\eta, \eta\} = p_4 y^2 \qquad \{y, \eta\} = p_3 y\eta \tag{1b}$$

where $p_1, \ldots, p_4$ are arbitrary even constants; as usual, latin and greek letters denote even and odd elements respectively. Taking

$$M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$$

as a general element of $\mathrm{Mat}(1 \mid 1)$, we demand that the relations (1) are preserved under the action of M (by multiplication) on the vectors $\begin{pmatrix} x \\ \xi \end{pmatrix}$ and $\begin{pmatrix} \eta \\ y \end{pmatrix}$

$$\begin{pmatrix} \overline{x} \\ \overline{\eta} \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} \qquad \begin{pmatrix} \overline{\eta} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} \eta \\ y \end{pmatrix}. \tag{2}$$

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This requirement uniquely determines a multiplicative (pre) Poisson structure on Mat(1 | 1)

$$\begin{aligned} &\{a,d\} = -(p_1+p_3)\beta\gamma && \{a,\beta\} = p_3 a\beta - p_4 d\gamma \\ &\{a,\gamma\} = p_1 a\gamma - p_2 d\beta && \{\beta,\beta\} = p_4(d^2-a^2) \\ &\{\gamma,\gamma\} = p_2(a^2-d^2) && \{\beta,\gamma\} = (p_1-p_3)\beta\gamma \\ &\{d,\beta\} = p_3 d\beta - p_4 a\gamma && \{d,\gamma\} = p_1 d\gamma - p_2 a\beta. \end{aligned} \tag{3}$$

This structure has the following properties:

(A) The superdeterminant

$$\mathrm{sdet}(M) = (d-\gamma a^{-1}\beta)a^{-1} \tag{4}$$

is central for all values of the parameters $p_1, \ldots, p_4$.

(B) The Poisson brackets (3) on Mat(1 | 1) satisfy the (graded) Jacobi identities *iff* the superplane Poisson brackets (1) do, which happens *iff*

$$p_1 p_2 = p_3 p_4 = 0. \tag{5}$$

Thus, there exist precisely 3 non-isomorphic families of multiplicative Poisson structures on Mat(1 | 1)

$$p_1 = p_3 = 0 \tag{6a}$$

$$p_2 = p_4 = 0 \tag{6b}$$

$$\{p_1 = p_4 = 0\} \text{ isomorphic to } \{p_2 = p_3 = 0\}. \tag{6c}$$

(C) The Poisson centre of Mat(1 | 1) is generated by

$$P(a,d) - \frac{dP_{,a} + aP_{,d}}{a^2-d^2}\gamma\beta \qquad \forall P \in C(a,d) \tag{7a}$$

for the case (6*a*); sdet(M) for the case (6*b*) when $p_1 + p_3 \neq 0$. If, in addition, $p_1 + p_3 = 0$, then the Poisson generators are

$$P(a/d) \text{ and } Q(a,d)\gamma\beta \qquad \forall P, Q \tag{7b}$$

$$P(a/d) - a^{-1}P_{,d}\gamma\beta \qquad \forall P \tag{7c}$$

for the case (6*c*) when $p_1 = p_4 = 0$, $p_3 \neq 0$. If $p_3 = 0$, then the generators are as in (7*a*).

Now let us turn to the quantum picture.

We start with the case (6*c*), $p_1 = p_4 = 0$. The quantum analogue of formulae (1) is

$$x\xi = \xi x \qquad \xi^2 = hx^2 \qquad \eta^2 = 0 \qquad y\eta = q\eta y. \tag{8}$$

The quantum matrix $M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$ preserves these relations *iff*

$$\begin{aligned} &a\beta = q\beta a \qquad a\gamma = \gamma a - h(1+q)\beta d \\ &ad = \tfrac{1}{2}(1+q^{-1})da + \tfrac{1}{2}(1-q^{-1})\gamma\beta \qquad d\beta = q\beta d \\ &d\gamma = \gamma d - h(1+q)\beta a \qquad \beta^2 = 0 \\ &\beta\gamma = \tfrac{1}{2}(q^{-1}-1)da - \tfrac{1}{2}(q^{-1}+1)\gamma\beta \qquad \gamma^2 = h(a^2 - d^2). \end{aligned} \tag{9}$$

Ordering monomials according to the rule $a > d > \beta > \gamma$ and using the diamond lemma (Bergman 1978) we find that formulae (9) do not have the PBW property for $q \neq 1$; e.g.

$$0 = \beta\beta\gamma = \tfrac{1}{4}(q^{-1}-1)^2(2q+q^2)\beta da.$$

The quantum version of the property (B) above breaks down here: it is easy to check that the quantum superplane

$$x\xi = q\xi x \qquad \xi^2 = hx^2$$

has the PBW property *iff*

$$h(q^2-1) = 0 \tag{10}$$

an analogue of formula (5), and this criterion is obviously satisfied by the superplanes (8). This is a rare instance of a non-quantizable situation.

Remark. This example answers in the negative the open problem 1.1 in Drinfel'd (1992), of whether every Lie bialgebra can be quantized. Drinfel'd proves that this is always possible modulo $(q-1)^4$.

Next we consider the case (6*b*), $p_2 = p_4 = 0$. Its quantum version is the pair of superplanes

$$x\xi = q^{-1}\xi x \qquad \xi^2 = 0 \qquad \eta^2 = 0 \qquad \eta y = Qy\eta. \tag{11}$$

The PBW criterion (10) is satisfied for both superplanes. The quantum matrix $M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$ is a symmetry of (11) *iff*

$$\begin{aligned} &\gamma^2 = 0 \qquad \gamma\beta = -qQ^{-1}\beta\gamma \qquad \gamma d = qd\gamma \qquad \gamma a = qa\gamma \\ &\beta^2 = 0 \qquad \beta d = Qd\beta \qquad \beta a = Qa\beta \\ &da = ad + (Q^{-1} - q)\beta\gamma. \end{aligned} \tag{12}$$

By construction, formulae (12) are multiplicative; we write $M \in \mathrm{Mat}_{q,Q}(1|1)$ for brevity. This time the PBW property holds true. Further, the quantum determinant $\mathrm{sDet}(M)$, given by the *same* classical formula (4), is central and multiplicative (i.e. group-like). One gets the quantum supergroups $GL_{q,Q}(1|1)$ and $SL_{q,Q}(1|1)$ thereby. For $Q = q$, these supergroups are known (Corrigan *et al* 1990) under the name $GL_q(1|1)$; for $Q \neq q$, they are discussed in Dabrowski and Wang (1991). Finally, using formulae

$$\mathrm{sDet}(M) = a^{-1}(d - \gamma a^{-1}\beta) = [d^{-1}(a - \beta d^{-1}\gamma)]^{-1} = [(a - \beta d^{-1}\gamma)d^{-1}]^{-1} \tag{13}$$

one can show that

$$M^{-1} = a^{-1}\begin{pmatrix} d + \beta a^{-1}\gamma - \gamma a^{-1}\beta & -\beta \\ -\gamma & a \end{pmatrix} a^{-1}[\mathrm{sDet}(M)]^{-1} \tag{14}$$

$$M^{-1} \in \mathrm{Mat}_{q^{-1},Q^{-1}}(1|1) \tag{15}$$

and that

$$\mathrm{sDet}(M^{-1}) = [\mathrm{sDet}(M)]^{-1}. \tag{16}$$

For $Q \neq q^{-1}$, one also has the following purely quantum formula:

$$\mathrm{sDet}(M) = \frac{qda^{-1} - Q^{-1}a^{-1}d}{q - Q^{-1}}. \tag{17}$$

In analogy with the case of $GL_q(1|1)$ (Schwenk *et al* 1990) one should expect that

$$M^k \in \mathrm{Mat}_{q^k,Q^k}(1|1) \qquad \mathrm{sDet}(M^k) = [\mathrm{sDet}(M)]^k \qquad \forall k \in \boldsymbol{Z}. \tag{18}$$

Suppose $L = \begin{pmatrix} X & \Phi \\ \Psi & Y \end{pmatrix}$ is a 2×2 Z_2-graded even matrix whose elements supercommute with those of M. Define

$$\mathrm{sTr}(L) = X - Y. \tag{19}$$

Then

$$\mathrm{sTr}(MLM^{-1}) = \mathrm{sTr}(M^{-1}LM) = \mathrm{sTr}(L). \tag{20}$$

Let us now turn to the last, and the most complicated, case (6*a*), $p_1 = p_3 = 0$. The quantum version of a pair of superplanes (1) is

$$x\xi = \xi x \qquad \xi^2 = h_1 x^2 \qquad \eta^2 = h_2 y^2 \qquad y\eta = \eta y \tag{21}$$

where h_1 and h_2, like $h, \ln q$ and $\ln Q$ before them, are the deformation parameters. The PBW criterion (10) is clearly satisfied. The quantum matrix $M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$ preserves the relations (21) *iff*

$$\begin{aligned}
&\gamma^2 = \frac{h_1}{1-\Delta}(a^2 - d^2) \qquad \gamma\beta = -\beta\gamma \qquad \Delta := h_1 h_2 \neq 1 \\
&\gamma d = \frac{1}{1-\Delta}[2h_1 a\beta + (1+\Delta)d\gamma] \\
&\gamma a = \frac{1}{1-\Delta}[(1+\Delta)a\gamma + 2h_1 d\beta] \qquad \beta^2 = \frac{h_2}{1-\Delta}(d^2 - a^2) \\
&\beta d = \frac{1}{1-\Delta}[2h_2 a\gamma + (1+\Delta)d\beta] \\
&\beta a = \frac{1}{1-\Delta}[(1+\Delta)a\beta + 2h_2 d\gamma] \qquad da = ad.
\end{aligned} \tag{22}$$

We write $M \in \mathrm{Mat}_{\hbar_1,\hbar_2}(1|1)$. These formulae have the following properties:

(A′) The expressions (4) and (13), in the form

$$\mathrm{stet}_1(M) = (d - \gamma a^{-1}\beta)a^{-1} = a^{-1}(d - \gamma a^{-1}\beta) \tag{23a}$$

and

$$\mathrm{stet}_2(M) = d^{-1}(a - \beta d^{-1}\gamma) = (a - \beta d^{-1}\gamma)d^{-1} \tag{23b}$$

are *central but not multiplicative*, and are no longer inverse to each other. Thus, the triangular decomposition methods for defining quantum superdeterminants (Kupershmidt 1990)

$$\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}\gamma & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d - \gamma a^{-1}\beta \end{pmatrix}\begin{pmatrix} 1 & a^{-1}\beta \\ 0 & 1 \end{pmatrix} \tag{24a}$$

$$= \begin{pmatrix} 1 & \beta d^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a - \beta d^{-1}\gamma & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d^{-1}\gamma & 1 \end{pmatrix} \tag{24b}$$

fail here. For the inverse matrix we have formula (14), with $\mathrm{stet}_1(M)$ substituted in place of $\mathrm{sDet}(M)$, as well as

$$M^{-1} = d^{-1}\begin{pmatrix} d & -\beta \\ -\gamma & a + \gamma d^{-1}\beta - \beta d^{-1}\gamma \end{pmatrix} d^{-1}[\mathrm{stet}_2(M)]^{-1}. \tag{25}$$

(B′) The PBW property holds true for $\mathrm{Mat}_{\hbar_1,\hbar_2}(1|1)$;

(C′) The element $a^2 - d^2$ is central, and so are β^2 and γ^2. We also have

$$[a, \beta\gamma] = [d, \beta\gamma] = 0 \tag{26}$$

so some analogue of formula (7*a*) may hold true also as a quantum version, since $[a, d] = 0$;

(D′) We still have to find a quantum superdeterminant. Recall why the difficulties arise only in the Z_2-graded case. In the non-Z_2-graded situation of linear algebra, if $A : V \to V$ is a linear operator on a finite-dimensional vector space (or a free module), its induced action on $\Lambda^n(V)$, $n = \dim V$, is an operator of multiplication by a scalar, since $\Lambda^n(V)$ is one-dimensional. This scalar, called the determinant of A, is by construction multiplicative. If V is now Z_2 graded, $V = V_0 + V_1$, there are no one-dimensional analogues of $\Lambda^n(V)$, so there is no analogue of the usual determinant. The new animal, $\mathrm{sdet}(A)$, the superdeterminant, is a *rational* function of A, and there is no immediately obvious one-dimensional module on which the $\mathrm{sdet}(A)$ acts. However, there exists something very close, and in any case sufficient for our purposes. Suppose $U \subset W$ is a pair of finite-dimensional vector spaces (or free modules) such that $\dim(W/U) = 1$, and suppose $A :\ W \to W$ is a linear operator for which U is invariant: $A(U) \subset U$. Then A acts on the one-dimensional space W/U, hence it acts by multiplication, and one obtains a multiplicative determinant. If $W = V_0 + V_1$ is Z_2-graded, then, for a basis $\{e_i\}$ of W, we set

$$W/U = \langle [\sqcap_i\{e_i | e_i \text{ is odd }\}][\sqcap_j\{e_j | e_j \text{ is even }\}]^{-1} \rangle. \tag{27}$$

For A even, we can reverse the Z_2-gradings of the basis $\{e_i\}$ and obtain a second version of the superdeterminant. The reader can check that in this way one the gets non-quantum formulae (23) and the usual two expressions for the superdeterminant. Let us see how this device works in our quantum case. We have, by (2) and (21)

$$\begin{aligned}\overline{\xi}\overline{x}^{-1} &= (\gamma x + d\xi)(ax + \beta\xi)^{-1} = (\gamma + d\xi x^{-1})(a + \beta\xi x^{-1})^{-1} \\ &= (\gamma + d\xi x^{-1})(1 - a^{-1}\beta\xi x^{-1})(a + h_1\beta a^{-1}\beta)^{-1} \\ &= \left[(\gamma + da^{-1}\beta h_1) + (d - \gamma a^{-1}\beta)\xi x^{-1}\right](a + h_1\beta a^{-1}\beta)^{-1}\end{aligned} \tag{28a}$$

and hence we get the first desired candidate for the quantum superdeterminant

$$\begin{aligned}\mathrm{sDet}_1(M) &= (d - \gamma a^{-1}\beta)(a + h_1\beta a^{-1}\beta)^{-1} \\ &= \mathrm{stet}_1(M)(1 + h_1\beta a^{-1}\beta a^{-1})^{-1}.\end{aligned} \tag{29a}$$

By construction, $\mathrm{sDet}_1(M)$ is multiplicative. One can check that $\beta a^{-1}\beta a^{-1}$ is central. Hence, $\mathrm{sDet}_1(M)$ is central as well. Similarly, we have

$$\begin{aligned}\overline{y}^{-1}\overline{\eta} &= (\gamma\eta + dy)^{-1}(a\eta + \beta y) = (\gamma y^{-1}\eta + d)^{-1}(ay^{-1}\eta + \beta) \\ &= (d + h_2\gamma d^{-1}\gamma)^{-1}(1 - \gamma d^{-1}y^{-1}\eta)(ay^{-1}\eta + \beta) \\ &= (d + h_2\gamma d^{-1}\gamma)^{-1}[(a + \gamma d^{-1}\beta)y^{-1}\eta + (\beta - \gamma d^{-1}ah_2)]\end{aligned} \tag{28b}$$

so that we get the second candidate for the quantum superdeterminant

$$\begin{aligned}\mathrm{sDet}_2(M) &= (d + h_2\gamma d^{-1}\gamma)^{-1}(a + \gamma d^{-1}\beta) \\ &= (1 + h_2 d^{-1}\gamma d^{-1}\gamma)^{-1}\mathrm{stet}_2(M).\end{aligned} \tag{29b}$$

Again, $\mathrm{sDet}_2(M)$ is multiplicative by construction. Also, $d^{-1}\gamma d^{-1}\gamma$ can be shown to be central. Thus, $\mathrm{sDet}_2(M)$ is also central. Finally, one can verify that

$$\mathrm{sDet}_1(M)\ \mathrm{sDet}_2(M) = 1. \tag{30}$$

The quantum supergroup $SL_{h_1,h_2}(1\,|\,1)$ is defined thereby. Here also one should except formulae analogous to (18) to hold.

Remark. In the non-Z_2-graded case of 2×2 matrices, one has two *one-parameter* families of quantum groups, $GL_q(2)$ and $GL_h(2)$. The Z_2-graded case is, as we see, *less rigid:* here we have two *two-parameter* families, $GL_{q,Q}(1\,|\,1)$ and $GL_{h_1,h_2}(1\,|\,1)$.

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